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Amplitude-independent chaotic synchronization

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There has recently been speculation on whether or not chaotic synchronization might be useful for communications. One problem with using chaotic synchronization to communicate is that the response system is nonlinear, so that any variation in the amplitude of the chaotic driving signal degrades the synchronization of the response system to the drive system. In the present work, I show that it is possible to design a response system that reproduces a scaled version of the chaotic driving signal when the drive signal is attenuated or amplified. A simple communications system is demonstrated to show that the type of synchronization described here is useful, and the effects of noise on the communications system are studied.

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I. INTRODUCTION

The synchronizing of chaotic systems that are coupled by a one-way driving is a popular research topic, with much speculation on possible applications to communications [1–14] One problem in using a chaotic signal as an information carrier is the problem of amplitude distortion, or fading. Broadcasting a chaotic signal, for example, may result in some attenuation between transmitter and receiver. Since the response system that is driven by the transmitted chaotic signal is nonlinear, changing the amplitude of the chaotic driving signal with throw the response system out of synchronization. A linear response system would not be sensitive to the amplitude of the driving signal, but completely linear response systems cannot be cascaded [3,4], so linear systems require more than one signal to confirm synchronization.

The work described below shows how to build a simple chaotic system so that the response system is not sensitive to the amplitude of the driving signal, but is still nonlinear. I also show that amplitude-independent chaotic systems may still send and receive information.

II. THEORY OF SYNCHRONIZATION

The theory of the synchronization of chaotic systems is described in detail elsewhere [2], so only a brief description is included here. We begin with a dynamical system that may be described by the ordinary differential equation

$$\dot{\alpha}(t)=f(\alpha). \quad (1)$$

The system is then divided into two subsystems, $\alpha=(\beta,\chi)$;

$$\dot{\beta}=g(\beta,\chi), \quad (2)$$
$$\dot{\chi}=h(\beta,\chi),$$

where $\beta=(\alpha_1,\ldots,\alpha_m)$, $g=(f_1(\alpha),\ldots,f_m(\alpha))$, $\chi=(\alpha_{m+1},\ldots,\alpha_n)$, and $h=(f_{m+1}(\alpha),\ldots,f_n(\alpha))$. The division is truly arbitrary since the reordering of the α_i variables before assigning them to β, χ, g, and h is allowed.

A first response system may be created by duplicating a new subsystem χ' identical to the χ system, substituting the set of variables β for the corresponding β' in the function h, and augmenting Eqs. (2) with this new system, giving

$$\begin{aligned}\dot{\beta}&=g(\beta,\chi),\\ \dot{\chi}&=h(\beta,\chi),\\ \dot{\chi}'&=h(\beta,\chi').\end{aligned} \quad (3)$$

If all the Lyapunov exponents of the χ' system (as it is driven) are less than zero, then $\chi'-\chi\rightarrow 0$ as $t\rightarrow\infty$. The variable β is known as the driving signal.

One may also reproduce the β subsystem and drive it with the χ' variable [3,4], giving

$$\begin{aligned}\dot{\beta}&=g(\beta,\chi),\\ \dot{\chi}&=h(\beta,\chi),\\ \dot{\chi}'&=h(\beta,\chi'),\\ \dot{\beta}''&=g(\beta'',\chi').\end{aligned} \quad (4)$$

The functions h and g may contain some of the same variables. If all the Lyapunov exponents of the χ', β'' subsystem are less than 0, then $\beta''\rightarrow\beta$ as $t\rightarrow\infty$. The example of Eq. (4) is referred to as cascaded synchronization. Synchronization may be confirmed by comparing the driving signal β with the signal β''.

III. AMPLITUDE-INDEPENDENT SYNCHRONIZATION

In the cascaded synchronization example of Eq. (4), every part of the drive system is reproduced at least once. Since a chaotic drive system requires some nonlinearity, there must be at least one nonlinear element in the response system. Changing the amplitude of the driving signal β will usually destroy synchronization. In many types of communications systems, transmitting the driving signal attenuates it by an unknown amount. This attenuation will destroy synchronization in a cascaded chaotic system. Oppenheim *et al.* [15] remedy this problem by adaptively changing the amplitude

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of the driving signal. The work in the present paper describes a nonadaptive way to maintain synchronization of chaotic systems.

To maintain synchronization when the drive signal has been attenuated (or amplified), the response system must contain only scale-invariant nonlinearities, which are nonlinear functions $f(x)$ with the property $f(Ax)=Af(x)$. A class of nonlinear functions that have the scale-invariant property consists of piecewise linear functions that have their only breakpoint at 0. A piecewise linear function consists of two or more line segments.

In order to build a chaotic system, scale-invariant nonlinearities may not be enough. In numerical and circuit experiments, systems that contained an instability and no scale dependent nonlinearity gave rise to unbounded motion; i.e., at least one variable increased towards $\pm\infty$. An amplitude-dependent nonlinear function was necessary to fold the motion back into a bounded region. The amplitude dependence may or may not be an actual requirement for a chaotic system, but it was found to be a requirement for the systems studied here.

Since the nonlinear folding function was amplitude dependent, it could not be included in the response system, so a full cascaded response system (which reproduces all parts of the drive system) could not be built. On the other hand, designing the nonlinear folding function so that the nonlinear response system had some of the desirable properties of a cascaded response system was possible. The nonlinear folding function produced the signal u_d that drove the nonlinear response system. The use of a function of a variable to drive the response system was similar to the work of Kocarev and Parlitz [10], in which the driving signal may be some function of the variables of the chaotic drive system. In the present work, however, not all parts of the drive system were reproduced in the response system, and the work in [10] did not require amplitude-invariant nonlinearities. Since the nonlinear folding function is not reproduced in the response system, it would be possible to have a completely linear response system. A linear response system might not be desirable because it might be possible to reconstruct the variables in the drive system using a series of integrators. Using nonlinear functions in the response should make reconstructing the drive signals more difficult unless one had a copy of the drive system.

IV. CIRCUIT EXAMPLE

The chaotic circuit described below fulfills all of the requirements described in the preceding section. The nonlinear folding function is $g_1(y)$, while amplitude-invariant nonlinearities are provided by $g_2(x)$ and $g_3(y)$. The equations for the circuit are

$$\frac{dx}{dt}=-\alpha[0.05x+0.05g_1(y)+1.47z+0.1S_I],$$

$$\frac{dy}{dt}=-\alpha[-0.5x-0.44g_1(y)+0.147y],$$

$$\frac{dz}{dt}=-\alpha[-0.5g_2(x)+z-0.5w],$$

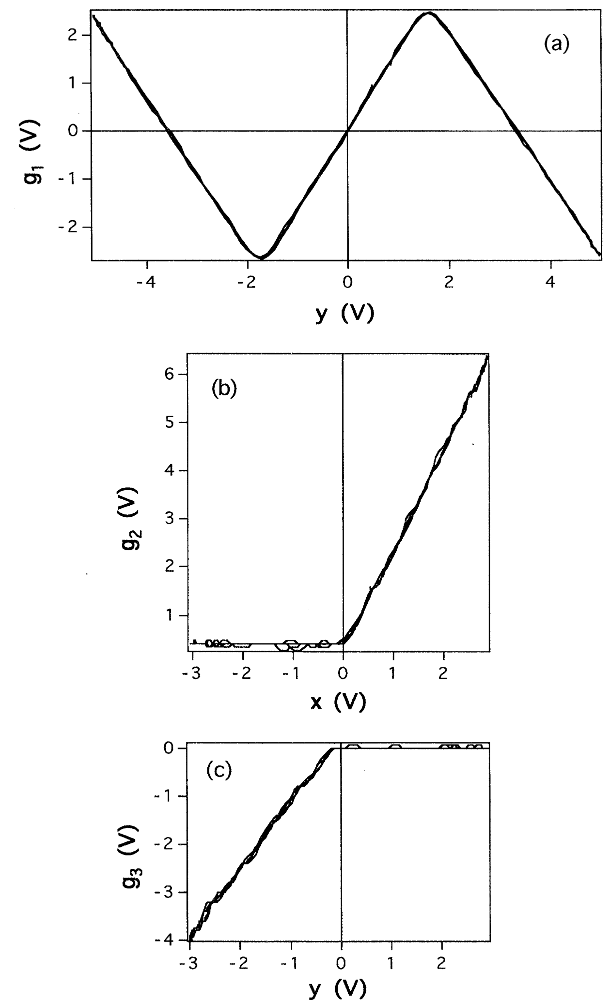

FIG. 1. (a) Graph of the function $g_1(y)$ created by driving this function in the circuit with a 10 Hz sine wave. The apparent noise is caused by digitization errors. (b) Graph of $g_2(x)$ from the circuit. (c) Graph of $g_3(y)$ from the circuit.

$$\frac{dw}{dt}=-\alpha[-10.0g_3(g_1(y))+10.0w],$$

$$g_1(y):\begin{cases} y\leqslant-1.6, & g_1=-2.5y-7.2,\\ -1.6<y<1.6, & g_1=2.0y,\\ 1.6\leqslant y, & g_1=-2.5y+7.2,\end{cases}$$

$$g_2(x):\begin{cases} x\leqslant 0, & g_2=0,\\ x>0, & g_2=4.5x,\end{cases} \tag{5}$$

$$g_3(y):\begin{cases} y\leqslant 0, & g_3=4.5y,\\ y>0, & g_3=0,\end{cases}$$

where the time factor $\alpha=10^4\ \mathrm{s}^{-1}$. S_I represents an information signal that may be injected into the circuit. Kocarev and Parlitz [10] also used signal injection method to encode information on a chaotic carrier. The piecewise linear functions $g_1(y)$, $g_2(x)$, and $g_3(y)$ are shown in Figs. 1(a), 1(b), and

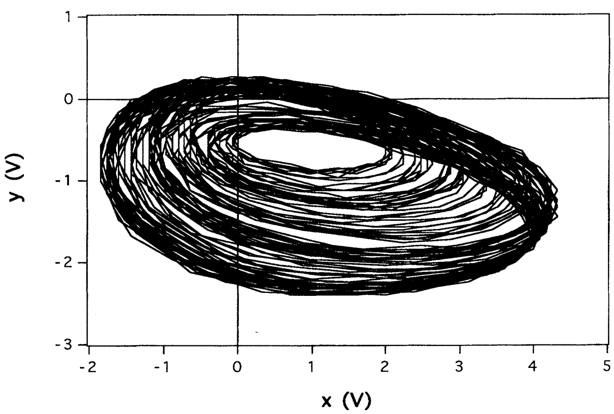

FIG. 2. Chaotic attractor from the circuit.

1(c). These plots were generated by driving each function in the circuit with a 10 Hz sine wave. Figure 2 is a chaotic attractor from the circuit described by Eq. (5). The largest Lyapunov exponent for this circuit was calculated numerically by the method of Eckmann and Ruelle [16] to be 765 s^{-1}.

A scaled version of $g_1(y)$ drove the response circuit. The response circuit equations were

$$u_d = A g_1(y),$$

$$\frac{dx'}{dt} = -\alpha(0.05x' + 0.05u_d + 1.47z'),$$

$$\frac{dy'}{dt} = -\alpha(-0.5x' - 0.44u_d + 0.147y'), \qquad (6)$$

$$\frac{dz'}{dt} = -\alpha[-0.5g_2(x') + z - 0.5w'],$$

$$\frac{dw'}{dt} = -\alpha[-10.0g_3(u_d) + 10.0w'],$$

where A is a scaling factor that may be greater or less than 1.0. The largest Lyapunov exponent for the response circuit in the synchronized state, calculated numerically from Eq. (6), was $-1470\ s^{-1}$, independent of the value of A.

Figure 3(a) confirms synchronization of the drive and response circuits by showing y' vs y from the circuit, for $A = 1.0$. When A is not 1, y' is a scaled version of y, as may be seen in Fig. 3(b), which shows y' vs y when $A = 0.2$. Figure 4 shows the attractor for the response circuit when $A = 0.5$. The Fig. 4 attractor is just a scaled version of the drive system attractor of Fig. 2.

V. COMMUNICATIONS

In order to send information from the drive to the response, it is not enough merely to synchronize the y and y' signals, because the y signal is not available at the response. At the response system, the drive signal u_d provides all the information about the state system, and this signal may be rescaled by an unknown amount. Comparing the drive signal u_d to the response output signal y' is enough to confirm synchronization between the drive and response systems if the nonlinear folding function $g_1(y)$ is chosen properly. The folding function can be chosen so that for some $y = y_0$, $A g_1(y_0) = y_0$. For the $g_1(y)$ defined in Eq. (5), $A g_1(y_0) = y_0$ is true for $y_0 = 0$. Figure 5 is a plot of y' vs u_d from the response circuit. The plot of Fig. 5 passes through the origin, at which point $y' = u_d$. Checking the value of u_d when y' crosses 0 tests for synchronization; if $u_d = 0$ at this time, the systems are synchronized. Comparing u_d to y' only at zero crossings of y' does limit the rate at which information may be transmitted to the average rate of 0 crossings, about 2 kHz for the circuit of Eq. (5).

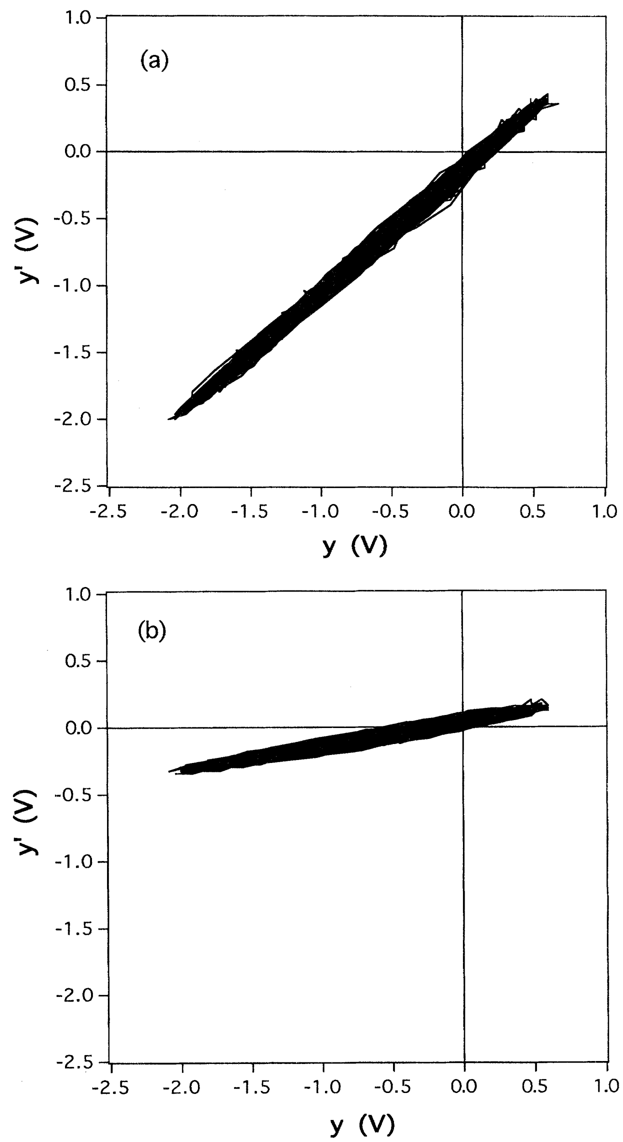

FIG. 3. (a) y' signal from the response circuit vs y signal from the drive circuit, showing synchronization, when the scaling factor A from Eq. (6) is 1.0. (b) y' signal from the response circuit vs y signal from the drive circuit when $A = 0.2$, showing that y' is a scaled down version of y.

There are many ways to encode information on a chaotic

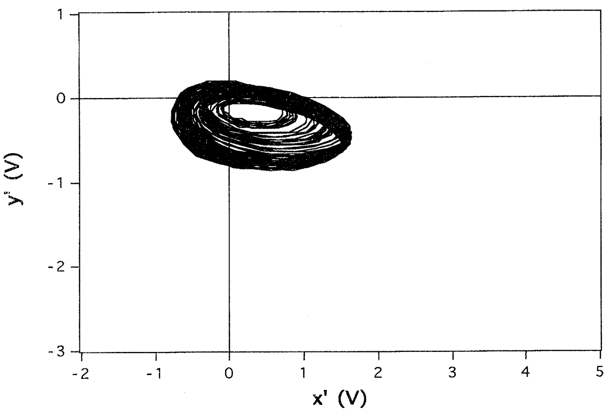

FIG. 4. Chaotic attractor from the response circuit when the scaling factor A from Eq. (6) is 0.5, showing that this attractor is a scaled down version of the drive circuit attractor in Fig. 2.

carrier. In the present work, the information is added into the dynamical system as the signal S_I in Eq. (5). For the circuit described by Eq. (5), adding information to a dynamical variable did not have a large effect on the amplitude of the chaotic output signal u_d.

An alternate method of encoding information would be to modulate a parameter in the driving circuit. The main effect of modulating a parameter in the drive circuit was to vary the amplitude of u_d. Because the response circuit was designed to be insensitive to changes in the amplitude of u_d, adding information to a dynamical variable was a more efficient way to encode a signal than parameter modulation. For other chaotic systems, parameter modulation may be more efficient. Kocarev and Parlitz [10] have noted that it is difficult to detect an injected information signal in the power spectrum of the drive signal, and the same effect is seen in the present work. The form of the information signal was $S_I = 2.0 \sin(2\pi f_I t)$, where f_I, the frequency of the information signal, ranged from 5 to 200 Hz. The largest negative Lyapunov exponent of the response system ($-1470\ \mathrm{s}^{-1}$), which governs how fast the response tracks a changing drive signal, limits the maximum information frequency.

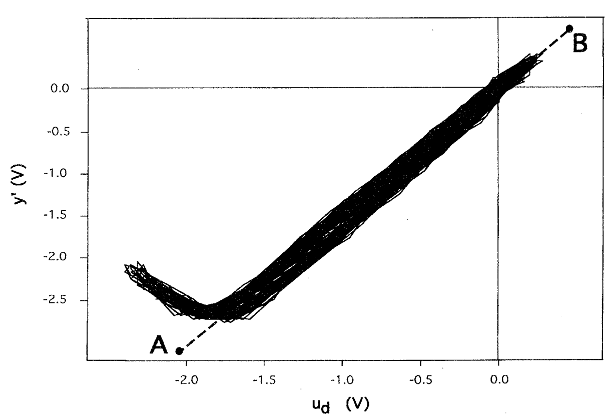

FIG. 5. Output signal y' from the response circuit vs the signal that drives the response circuit u_d.

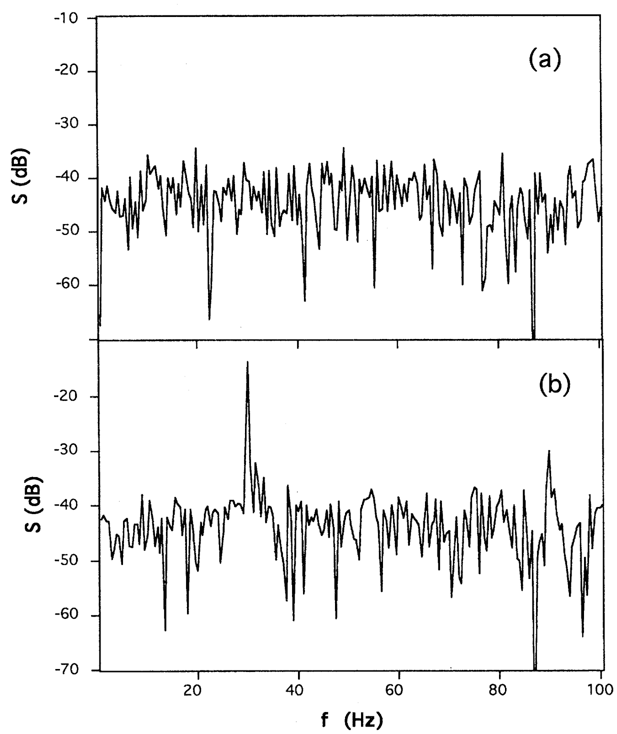

FIG. 6. (a) Low frequency portion of the power spectrum of the u_d signal when a 30 Hz information signal has been injected into the drive system as in Eq. (5). (b) Low frequency portion of the detected signal Δ created by strobing the input to the response circuit with the output, when a 30 Hz drive signal has been injected into the drive system. Note the improvement in the signal to noise ratio at 30 Hz.

In order to detect the information in the chaotic carrier signal u_d, the negative-going zero crossings of the signal y' from the response circuit were used to strobe u_d, generating a detected signal Δ. When the drive and response were synchronized, u_d was zero when y' was zero, so Δ was zero. When the drive and response were not synchronized, then u_d was nonzero when y' crossed zero, so Δ was nonzero. In general, Δ was a complicated time series. If some parameter in the drive system was changed by a small amount (or if a small signal was injected into the drive system), then the average of Δ was proportional to the parameter difference between the drive and response systems (or the average of Δ was proportional to the injected signal). If the information frequency was much lower than the average rate at which y' crossed zero, then Δ could also be used to detect the information signal, since the information signal caused a lack of synchronization. Passing Δ through a low pass filter (or a band pass filter) had the effect of averaging Δ to reveal the information signal. The signal to noise ratio for the detected information signal was measured by taking a Fourier transform of Δ and measuring the signal to noise ratio at the information frequency.

Figure 6(a) is the low frequency portion of the power spectrum for u_d when the scaling factor $A = 1.0$ and the information frequency $f_I = 30$ Hz. The signal to noise ratio at the information frequency was measured by subtracting the average signal power (in dB) within 2 Hz of f_I (not including f_I) from the signal power at f_I. The signal to noise ratio for Fig. 6(a) was 4 dB, almost 20 dB lower than the signal to noise ratio in the detected signal (below).

Figure 6(b) is the low frequency portion of the power spectrum of the detected signal Δ for $f_I=30$ Hz. The signal to noise ratio at the information frequency was 23 dB. When the scaling factor A was 0.5, the signal to noise ratio at the information frequency was still 22 dB. For a small enough A, the signal to noise ratio will degrade due to circuit mismatch and noise.

Injecting the information signal into the chaotic drive system appears to be a particularly efficient way of encoding information for detection by an amplitude-independent response system. For the circuit of Eq. (5), it is easy to see how different encoding methods affect the detection process. In Fig. 5, the point where the y' vs u_d curve crosses zero lies along the line AB. Injecting an information signal into the drive circuit makes the y' vs u_d curve move perpendicular to the line AB, so signal injection produces the largest possible change in the zero crossing points of y' and u_d. The main effect of modulating a parameter in the drive circuit is to stretch the y' vs u_d curve along the line AB. Because parameter modulation causes a change mostly along the line AB, y' and u_d still cross zero at almost the same time, producing only a small detected signal Δ. The effects of different types of signal encoding are system dependent; for the circuit described by Eq. (5), the fact that the different types of signal encoding move the plot of y' vs u_d in different directions might be useful for sending multiple information signals on one chaotic signal.

VI. NOISE AND FILTERING

In order to be useful for communications, the response circuit must not be too sensitive to noise. The two types of noise that were considered here were additive deterministic noise near the chaotic carrier frequency and low frequency additive noise.

The most difficult type of noise to separate from a chaotic carrier signal should be a signal from another chaotic system with a similar frequency spectrum. Figure 7(a) is the power spectrum of the chaotic driving signal u_d from the circuit of Eq. (5). Figure 7(b) is the power spectrum of a contaminating signal [the ψ signal in Eq. (7)] from a Rossler circuit [17] described by the equations

$$\frac{d\xi}{dt}=-\alpha(\Gamma\xi+\beta\psi+\lambda\zeta),$$

$$\frac{d\psi}{dt}=-\alpha(-\xi-\gamma\psi),$$

$$\frac{d\zeta}{dt}=-\alpha[-g(\xi)+\zeta], \tag{7}$$

$$g(\xi)=\begin{cases}0, & \xi\leq 3\\ \mu(\xi-3), & \xi>3,\end{cases}$$

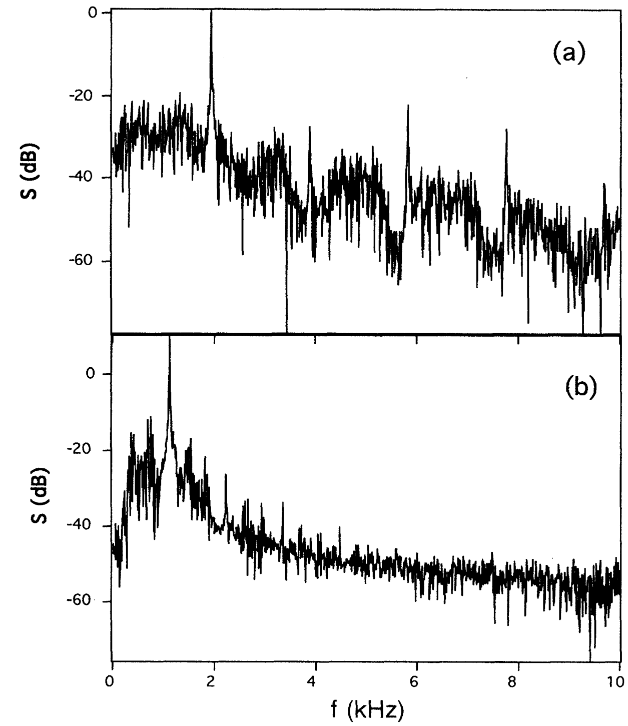

FIG. 7. Power spectrum of the u_d signal from the chaotic drive circuit described by Eq. (5). (b) Power spectrum of the ψ signal from the Rossler circuit described by Eq. (7). The ψ signal was added to u_d as a contaminating signal.

where the time factor $\alpha=10^4$ s^{-1}, $\Gamma=0.05$, $\beta=0.5$, $\lambda=1.0$, $\gamma=0.133$, and $\mu=15$. The ψ signal was added to u_d so that the rms amplitude of ψ was up to 1.4 times the rms amplitude of u_d (a signal to noise ratio of -1.5 dB).

When the contaminating signal ψ was added to u_d, an information frequency f_I of 10 Hz was used. Lower information frequencies improve the signal to noise ratio in the detected signal Δ because each cycle of the information signal is averaged over more zero crossings of y'. When f_I was 10 Hz and the transmitted signal u_d was 1.5 dB below the contaminating signal ψ, the signal to noise ratio of the information in the detected signal Δ was 15 dB (with no added noise the signal to noise ratio for Δ was 26 dB). The signal to noise ratio decreased rapidly for larger amplitudes of the contaminating signal. Recovery of the information signal is possible when a contaminating signal is present because the contaminating signal ψ is generated by a chaotic system that is not too similar to the drive system of Eq. (5). Part of the output signal y' from the response circuit is caused by ψ. This part of y' is not correlated with ψ (unless the contaminating signal is large enough to substantially alter the dynamics of the response circuit). The part of y' due to ψ will not contribute to the average of the detected signal Δ, so a low amplitude contaminating signal does not prevent information transmission. The same type of signal detection has been demonstrated before [18,19].

Low frequency noise has a more drastic effect on information recovery because it cannot be averaged away. Filtering can remove low frequency noise since the chaotic signal

u_d contains little power at low frequencies. An SR-560 preamp set to have a gain of 1 was used as a high pass filter that rolled off frequencies below 300 Hz at 12 dB/octave. The signal u_d (with an information frequency of 10 Hz) passed through the high pass filter before driving the response system. The high pass filter did not have much effect on u_d except that it removed a dc bias in u_d. The dc error could be corrected for in an adaptive fashion by adding a variable dc bias to u_d after the filter and adjusting the bias to optimize the match between y' and the filtered version of u_d. When the adaptive bias was used, no loss in signal to noise ratio for the filtered signal was seen; the ratio after filtering vs 27 dB, compared to a ratio of 26 dB without filtering. If no bias adjustment was used, the signal to noise ratio was 19 dB. Presumably the high pass filtering is possible because the injected information signal frequency is mixed with the other frequencies in the chaotic attractor, so that information is carried in a continuous band of intermodulation frequencies. An engineering analogy would be shifting up the frequency of an information signal by combining it in a nonlinear fashion with a higher frequency carrier. The chaotic response circuit then serves as a demodulator, recovering the information signal.

VII. CONCLUSIONS

Unknown amplitude variations in the amplitude of a chaotic driving signal need not be a problem when communicating with synchronized chaos. There are still other types of distortion that may disrupt chaotic communications signals, but the large variety of nonlinear systems that may be designed suggests that it may be possible to overcome other problems as well.

Amplitude-independent chaotic synchronization may be used to transmit information, even in the presence of large amounts of noise. Not only is communication in the presence of noise necessary for practical use, it also offers some advantage in shielding signals from eavesdroppers. It is possible to extract messages from chaotic signals by estimating some part of the message-free chaotic signal [20,21]. Short [20] was able to separate periodic signals from chaotic signals, but he found that it was harder to separate chaotic signals from other chaotic signals. Adding chaotic noise to a chaotic carrier signal might make estimating the chaotic system dynamics more difficult, making it harder to extract the message.

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